Driven Torsional Oscillator

Physics 401, Spring 2013

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(1)

Agenda

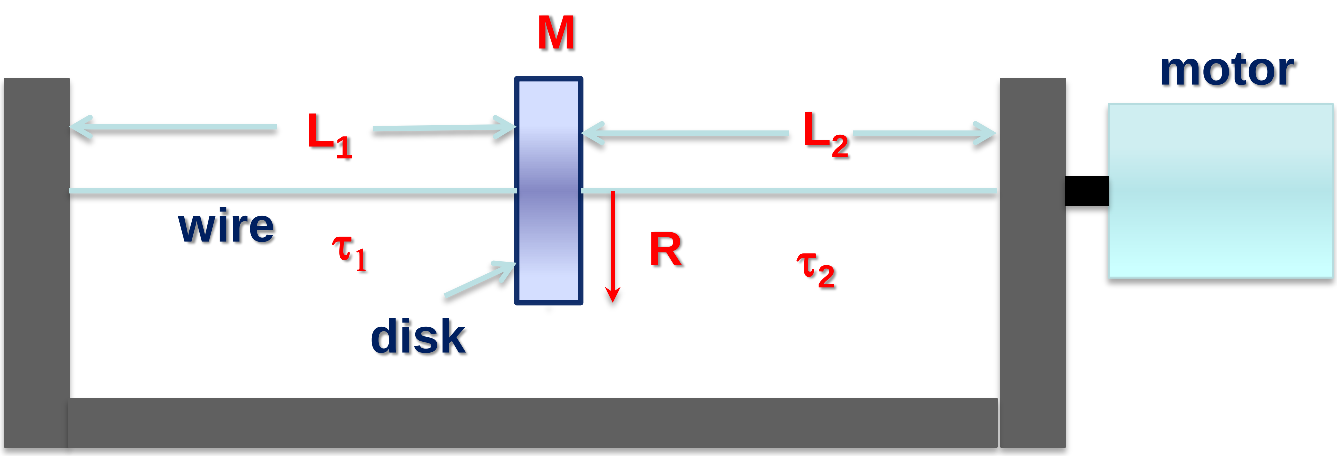
1. Driven torsional oscillator. Equations
2. Setup. Kinematics
3. Resonance
4. Beats
5. Nonlinear effects
6. Comments

Torsional oscillations. Resonance.

Tacoma (WA) Narrows Bridge, 1940

Driven torsional oscillator

The goals: (i) analyze the response of the damped driven harmonic oscillator to a sinusoidal drive. (ii) transient response and (iii) steady-state solution.



Angular displacement:

$$\theta_0 \cos(\omega t);$$

torque:

$$K\lambda\theta_0 \cos(\omega t)$$

$$\lambda = \frac{L_1}{L_1 + L_2}$$

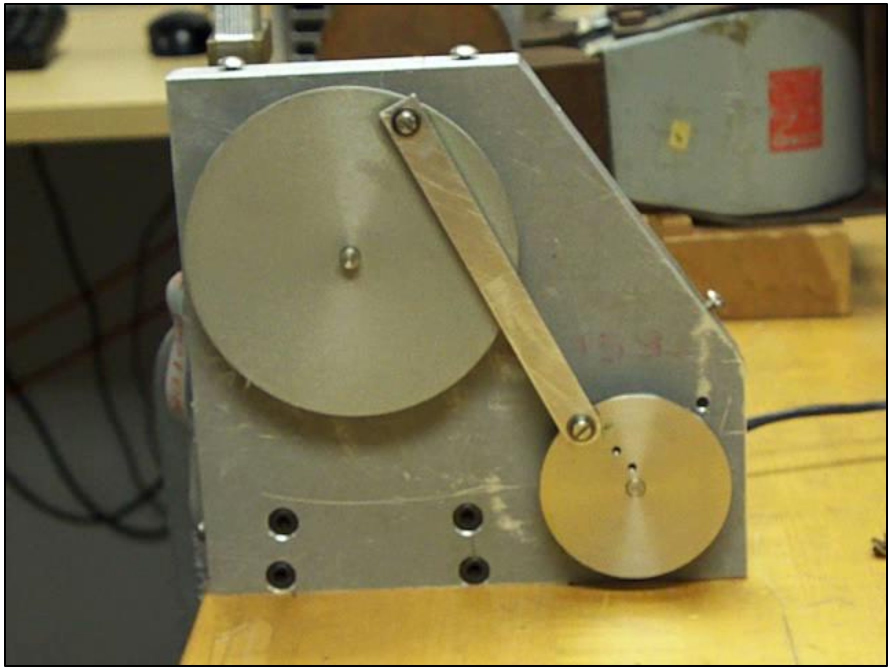
$$I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda\theta_0 \cos(\omega t)$$

Viscous damping

Torque by motor

I is momentum of inertia, [kg·m^2]
R is a damping constant [N·m·s].
K is the total spring constant [N·m]

Driven torsional oscillator



Motor

Pendulum

Transient solution

$$I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda\theta_0\cos(\omega t)$$

Solutions: sum of (1) Transient solution + (2) steady solution due to torque τ_m

(1) Transient solution (1st week experiment)

$I\ddot{\theta} + R\dot{\theta} + K\theta = 0$	The homogeneous equation of motion
$\theta(t) = Ae^{-at}\cos(\omega_1 t - \phi)$	Transient solution
$a = R/2I$	Attenuation constant
$\omega_o = \sqrt{K/I}$	Natural (angular) frequency
$\omega_1 = \sqrt{\omega_o^2 - a^2}$	Damped (angular) frequency

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Steady-state solution

$$\theta_t(t) = |A| e^{-at}\cos(\omega_1 t + \phi) \quad \rightarrow \quad \omega_1 = \sqrt{\omega_0^2 - a^2}$$ **Transient solution**

Once this response dies away in time the system response only on the frequency of drive ω

Initially the system responds on the characteristic frequency ω_1

So the steady-state solution must have the similar time dependence as the drive

$$\theta_{ss}(t) = \mathrm{Re}\left(\theta(\omega) e^{i\omega t}\right) \Rightarrow I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda\theta_0\cos(\omega t)$$

Substituting $\theta_{ss}(t)$ in equation of motion we will find the equations for $\theta(\omega)$

$$\theta(\omega) = \frac{\lambda\omega_0^2\theta_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 a^2}} e^{-i\beta(\omega)}$$

and

$$\beta(\omega) = \tan^{-1}\left(\frac{2\omega a}{\omega_0^2 - \omega^2}\right)$$

Steady-state solution. Summary.

$$I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda\theta_0\cos(\omega t)$$

(2) steady solution

$$\theta_s(t) = B(\omega)\cos\left(\omega t - \beta(\omega)\right)$$

Steady state solution

$$B(\omega) = \frac{\lambda\theta_o\,\omega_o^2}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2\gamma^2}}$$

Amplitude function

$$\tan\beta(\omega) = \frac{\omega\gamma}{\omega_o^2 - \omega^2}$$

Phase function

$$\gamma = \frac{R}{I} = 2\frac{R}{2I} = 2a$$

Damping constant

General solution

time domain form for steady-state solution will be

$$\theta_{ss}(t) = \frac{\lambda\omega_0^2\theta_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 a^2}} \cos(\omega t - \beta(\omega))$$

Amplitude $B(\omega)$

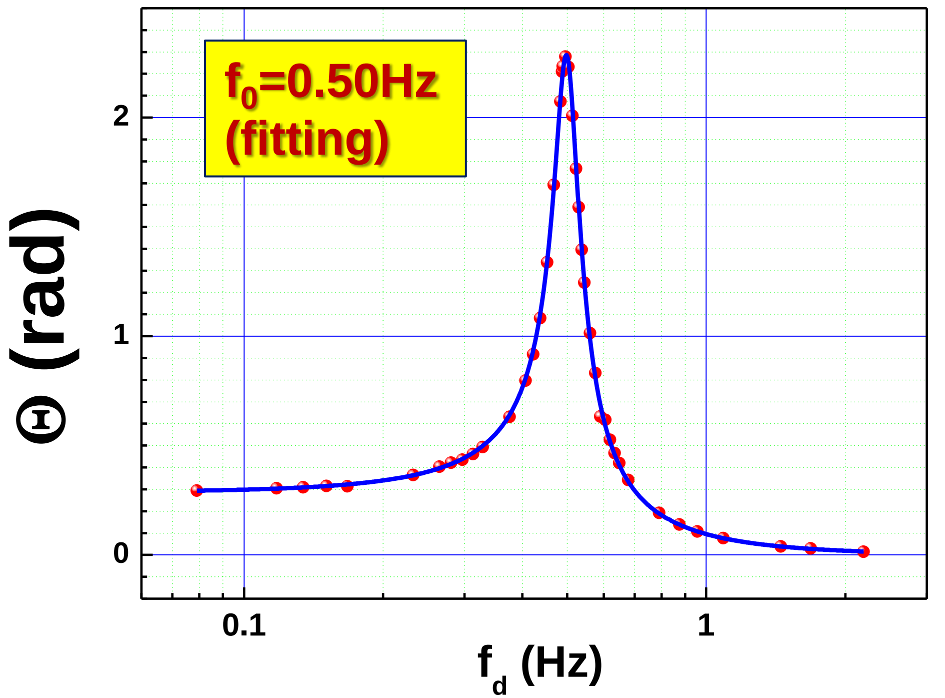
Phase

General solution for equation of motion consist of the sum of sum of two components: $\theta(t) = \theta_t(t) + \theta_{ss}(t)$

$$\theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at}\cos(\omega_1 t - \phi) + B\cos(\omega t - \beta(\omega))$$

Coefficients A and ϕ could be determined from initial conditions

Resonance. Experiment. Amplitude



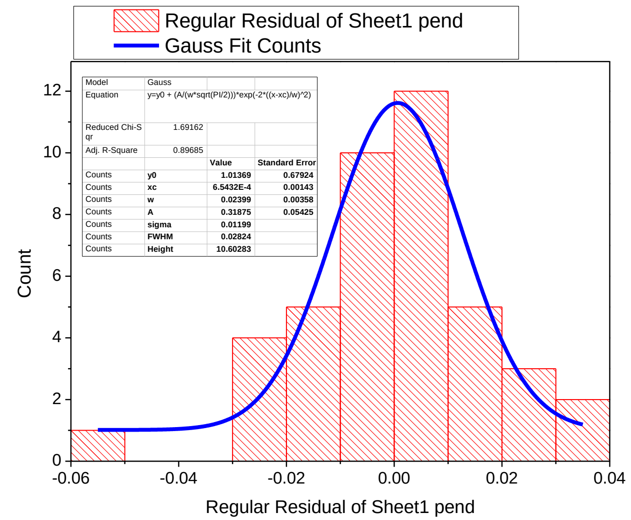
Model	Resonance1 (User)		
Equation	y=A*f0^2/sqrt((f0^2-x^2)^2+x^2*gamma^2)		
Reduced Chi-Sqr	3.00E-04		
Adj. R-Square	0.999411988		
		Value	Standard Error
pend	A	**0.286662**	0.001663551
pend	f0	**0.500271**	2.14E-04
pend	gamma	**0.062856**	4.98E-04

Fitting function:

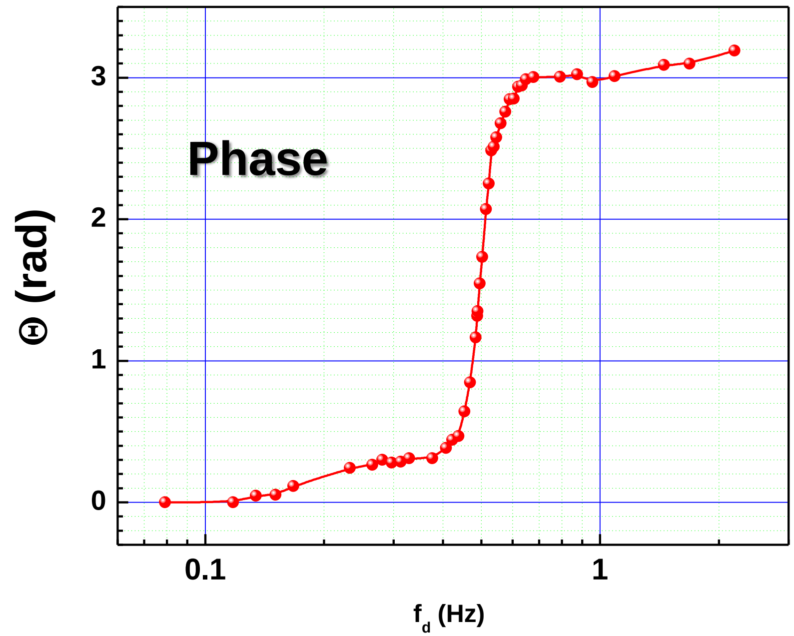
$$\theta(f) = \frac{A \bullet f_0^2}{\sqrt{\left(f_0^2 - f^2\right)^2 + \gamma^2 f^2}}$$

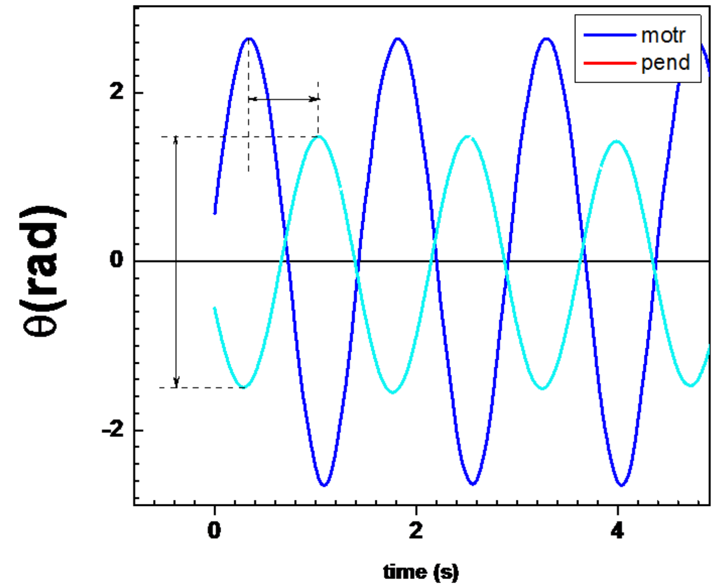
$\omega = 2\pi f$; $\gamma = 2a$

To create a new fitting function go "*Tools*"→"*Fitting Function Builder*" or press *F8*



Resonance. Experiment. Phase



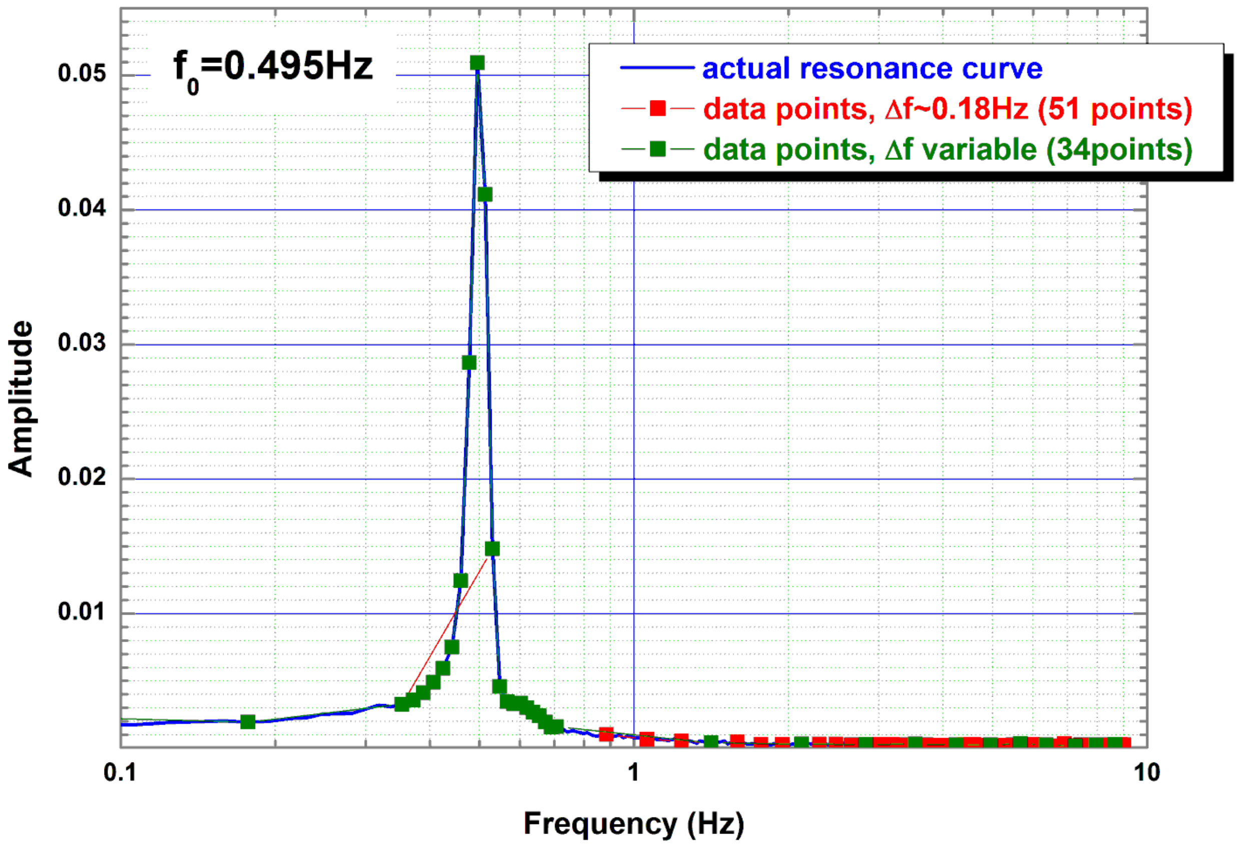


Scanning the driving frequency we can measure the amplitude of the pendulum oscillating and the phase shift

Both parameters Amplitude and phase can be defined by DAQ program or using Origin

Resonance. Experiment. Taking data.

For correct representation of the resonance curve take care about choosing of the step size in frequency.

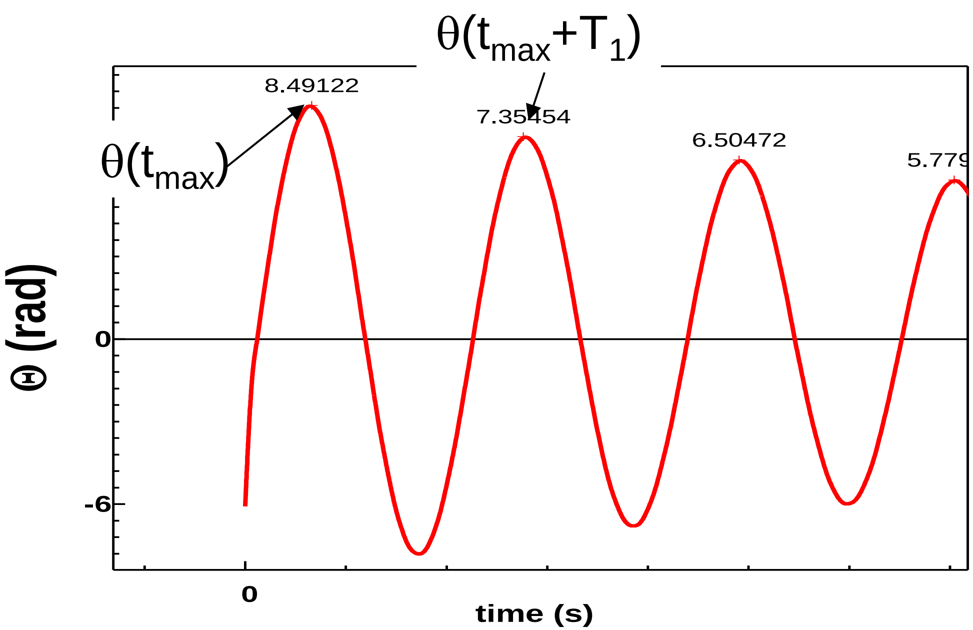


Quality factor and log decrement

There are two parameters used to measure the rate at which the oscillations of a system are damped out. One parameter is the logarithmic decrement δ, and the other is the quality factor, *Q*.

δ, is defined by

$$\delta = \ln\left(\frac{\theta(t_{\max})}{\theta(t_{\max}+T_1)}\right) = \ln\left(\frac{e^{-at_{\max}}}{e^{-a(t_{\max}+T_1)}}\right) = aT_1 .$$



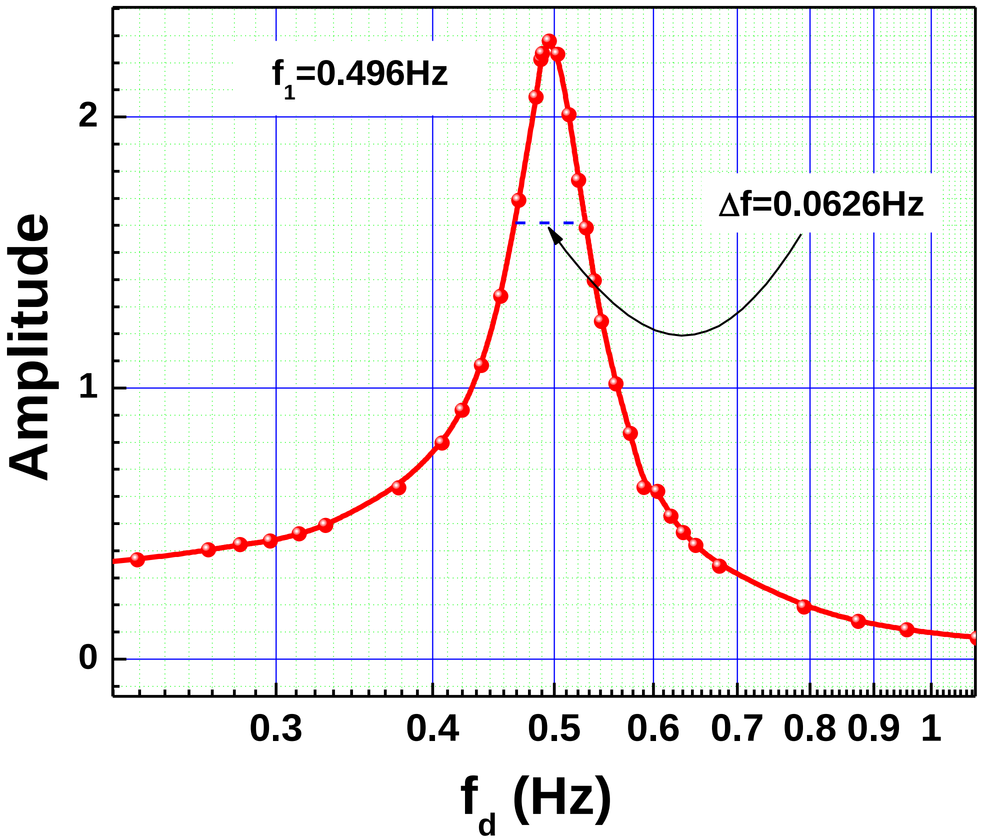
$$\delta = \ln\left(\frac{8.49}{7.35}\right) \approx 0.144$$

$$Q = 2\pi \frac{\text{total stored energy}}{\text{decrease in energy per period}} .$$

$$Q = \frac{\omega_1}{R/I} = \frac{\omega_1}{2a} = \frac{\pi}{a}\frac{\omega_1}{2\pi} = \frac{\pi}{a}\frac{1}{T_1} = \frac{\pi}{\delta}$$

$$Q \sim 21.8$$

Quality factor and log decrement



It can be shown that Q can be calculated as $\omega_1/\Delta\omega$ or $f_1/\Delta f$. $\Delta\omega$ is bandwidth of the resonance curve on the half power level or $\frac{\theta_{max}}{\sqrt{2}}$ for amplitude graph

Here *Q~7.9*

Beats. Theory.

Consider sum of two harmonic signals of frequencies ω_1 and ω_2

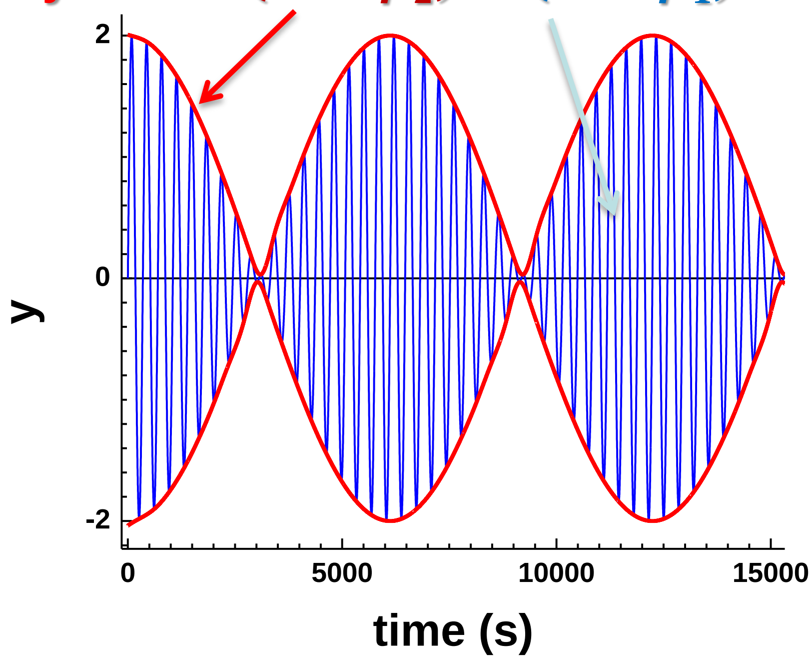
$y_1 = A\sin(\omega_1 t + \varphi_1)$; $y_2 = B\sin(\omega_2 t + \varphi_2)$

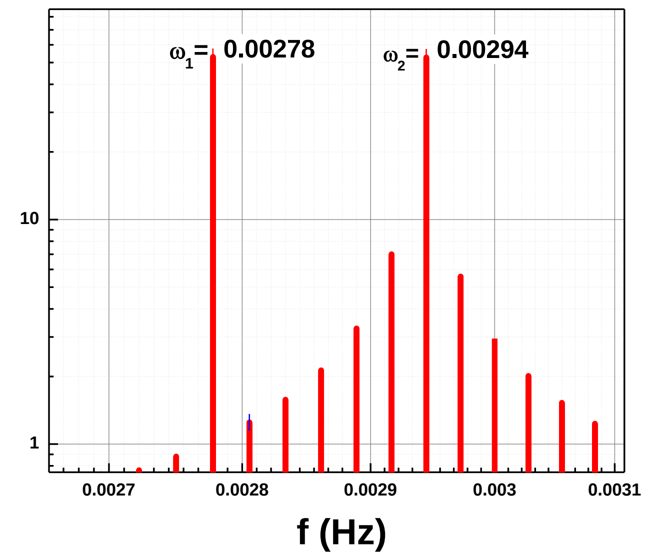
In case A=B $y = y_1 + y_2 = 2A\sin\left(\frac{\omega_1+\omega_2}{2}t + \beta_1\right)\cos\left(\frac{\omega_1-\omega_2}{2}t + \beta_2\right)$;

$\beta_1 = \frac{\varphi_1+\varphi_2}{2}$; $\beta_2 = \frac{\varphi_1-\varphi_2}{2}$

If $\omega_1 \approx \omega_2$ $\approx \frac{\omega_1+\omega_2}{2} = \omega$ and $\frac{\omega_1-\omega_2}{2} = \Omega$

$y = 2A\cos(\Omega t + \beta_2)\sin(\omega t + \beta_1)$





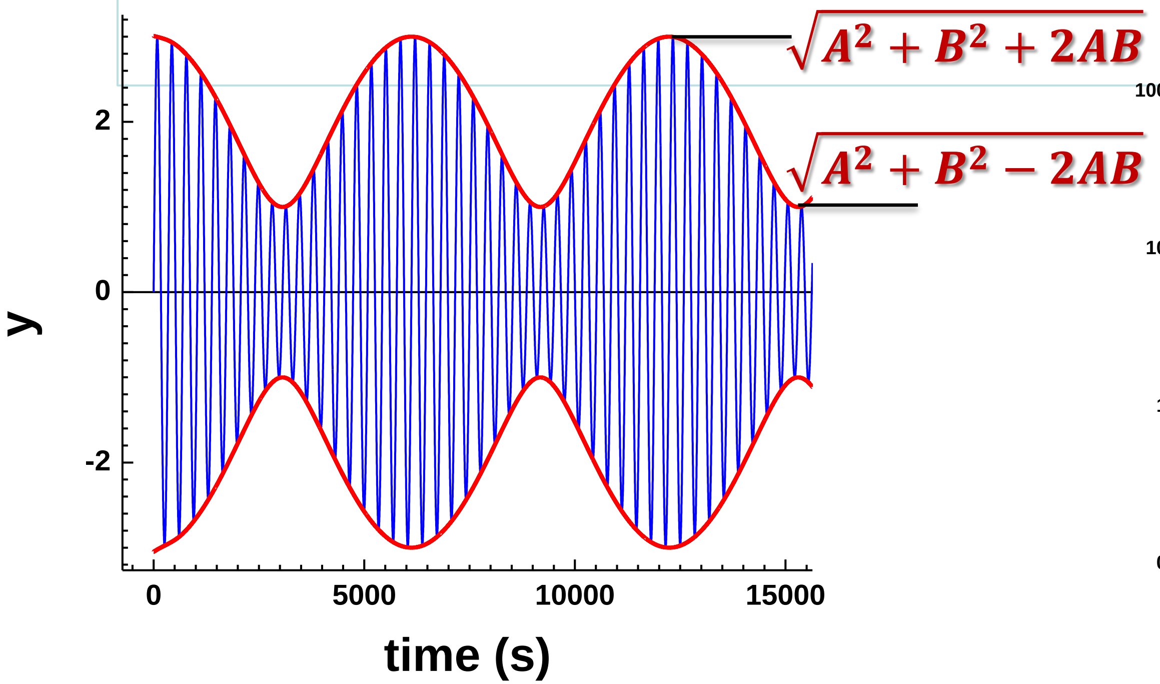
Beats. Theory.

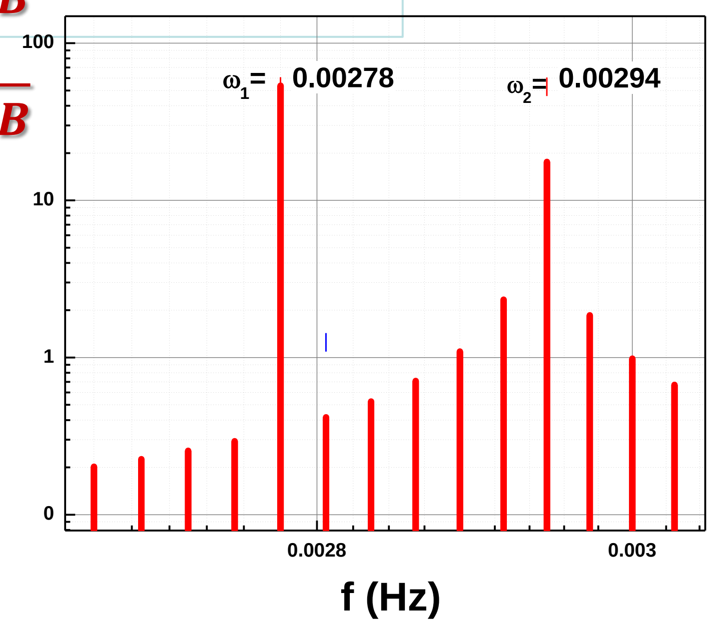
More general case A≠B ω_1 and ω_2

$y_1 = A\sin(\omega_1 t)$; $y_2 = B\sin((\omega_1+\alpha)t)$

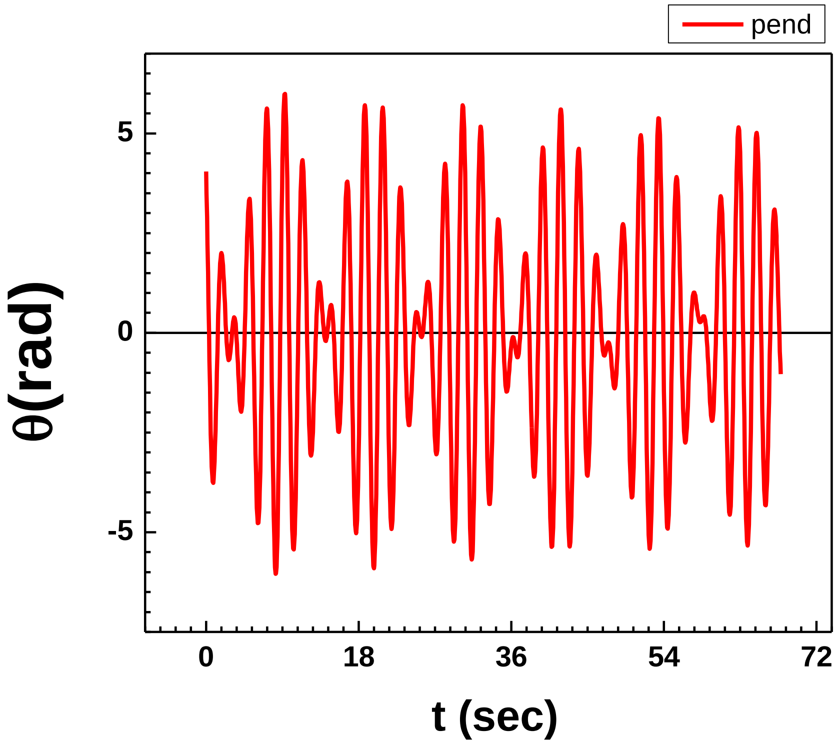
$y = y_1 + y_2 = C\sin((\omega+\beta)t)$ where $C = \sqrt{A^2 + B^2 + 2AB\cos(\alpha t)}$

$$\beta = \tan^{-1}\left(\frac{B\sin(\alpha t)}{A + B\cos(\alpha t)}\right) + \begin{cases} 0 & \text{if } A + B\cos(\alpha t) \geq 0 \\ \pi & \text{if } A + B\cos(\alpha t) < 0 \end{cases}$$

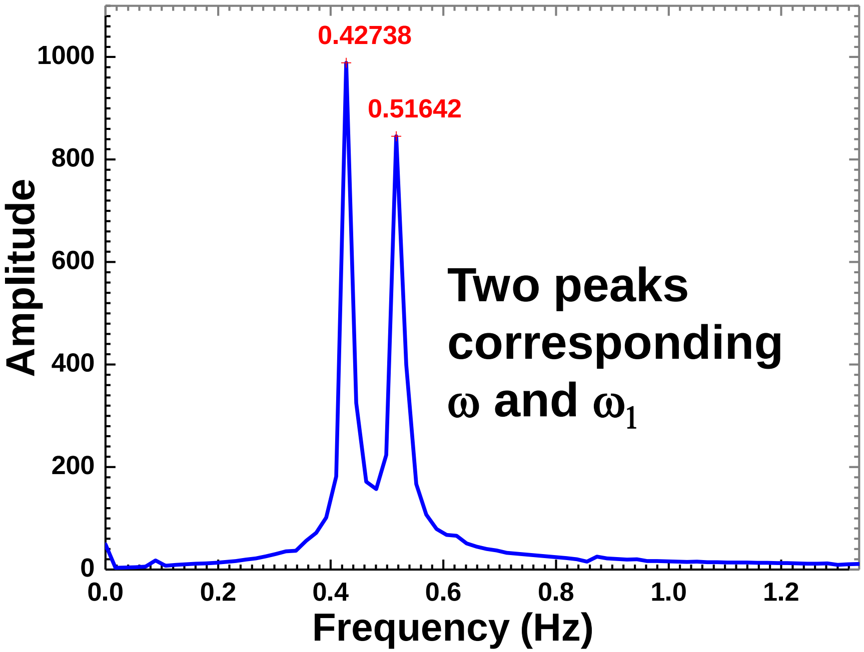




Beats. Experiment



Time domain trace



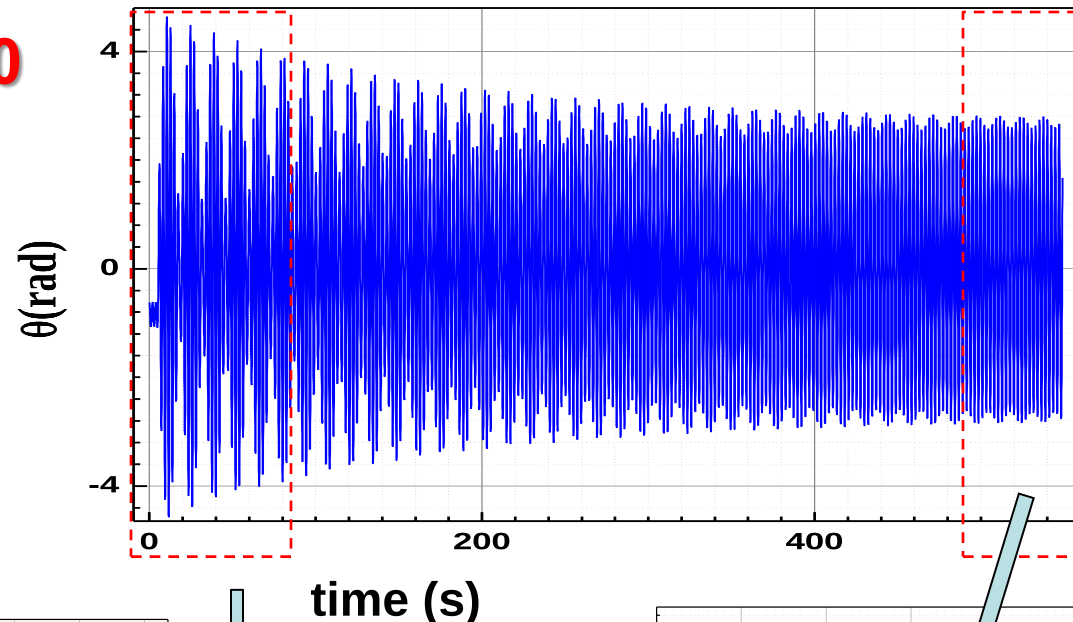
Beating spectrum

Use Origin to analyze the frequency spectrum !

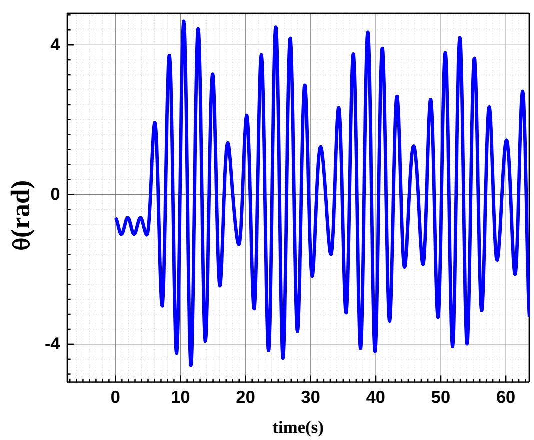
Beats. Experiment.

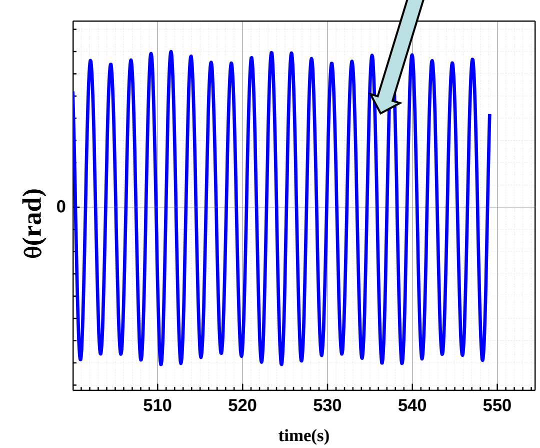
$$\theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at}\cos(\omega_1 t - \phi) + B\cos(\omega t - \beta(\omega))$$

$$\theta_t(t) \to 0$$



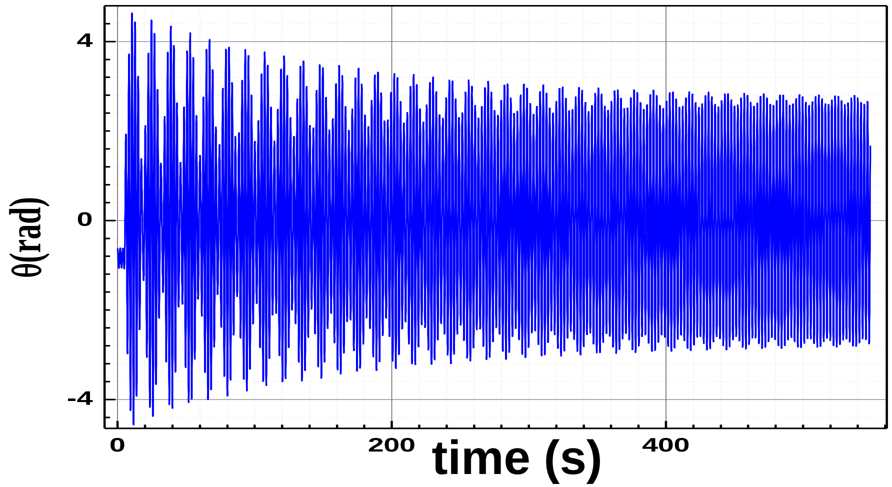
Beats dying in time. How fast – it depends on damping. When you will work on resonance data – wait until you will see the steady state oscillations.

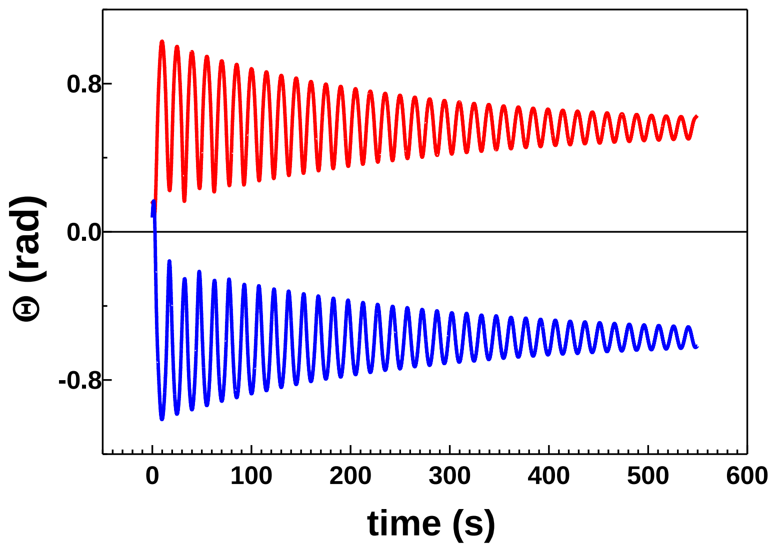




Beats. Experiment.

$$\theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at}\cos(\omega_1 t - \phi) + B\cos(\omega t - \beta(\omega))$$



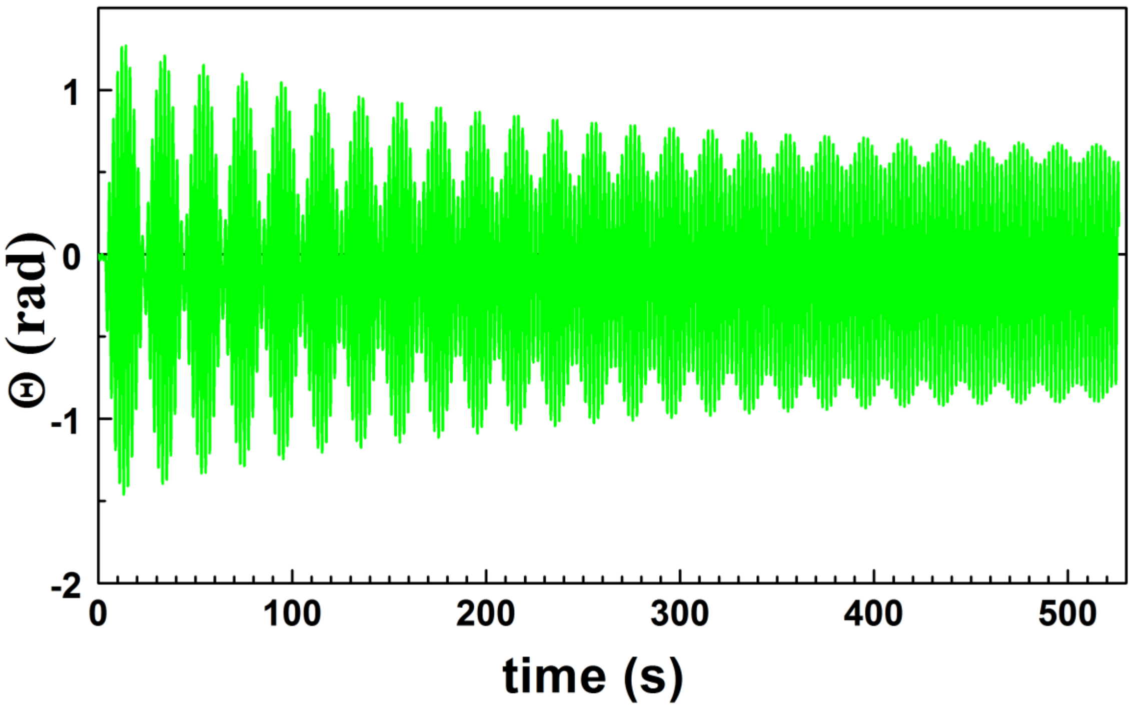


$\theta_t(t) \rightarrow 0$ **This can be seen well from "*envelope*" plot**

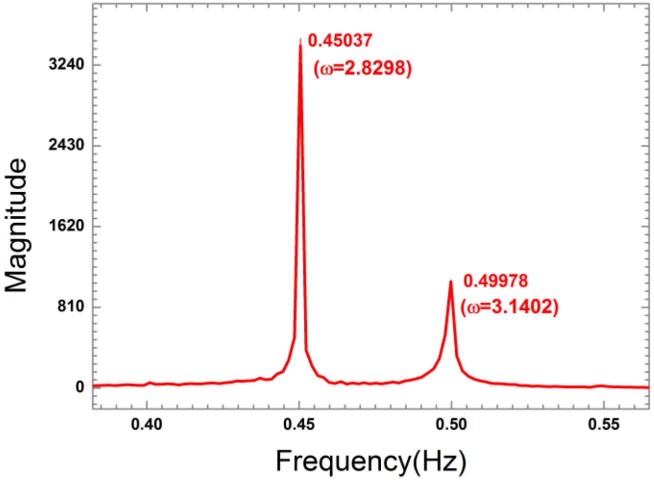
Origin 8.6: *Analysis → Signal Processing → Envelope*

Beats. Experiment. Fitting.

$$\theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at}\cos(\omega_1 t - \phi) + B\cos(\omega t - \beta(\omega)) + C$$



First let we apply FFT to find ω_1 and ω

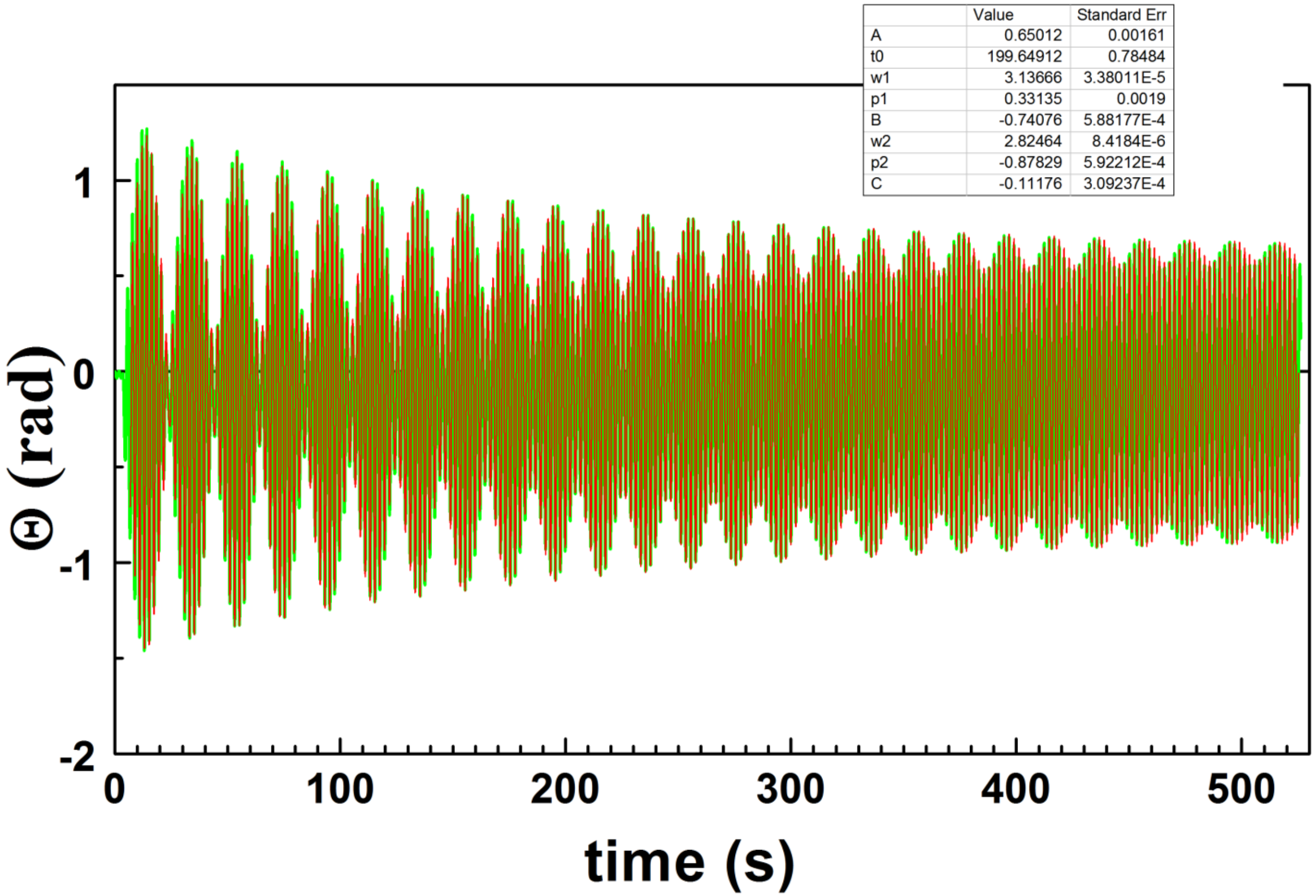


Result: ω_1=3.1402rad^{-1} and ω=2.8298 rad^{-1}

Beats. Experiment. Fitting.

$$\theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-\frac{t}{t_0}}\cos(\omega_1 t - \phi) + B\cos(\omega t - \beta(\omega)) + C$$

8 fitting parameters



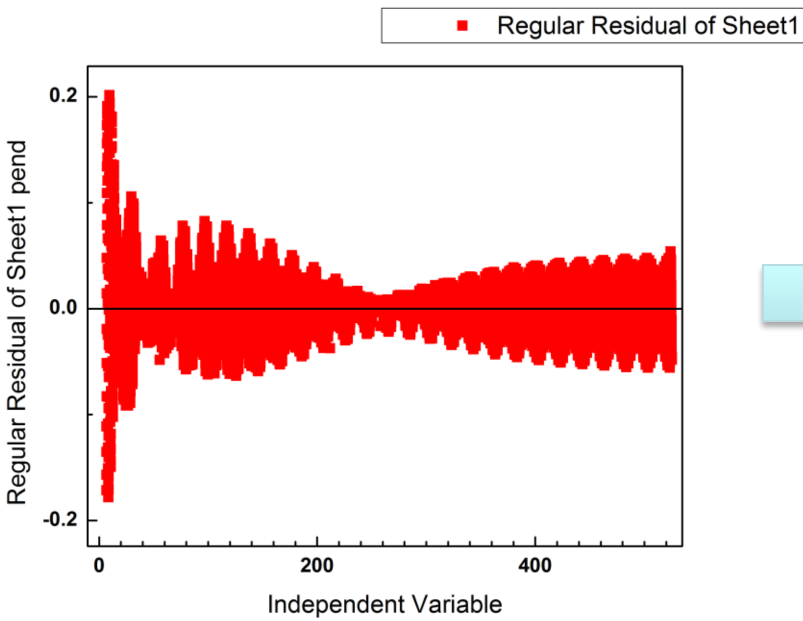
	Value	Standard Err
A	0.65012	0.00161
t0	199.64912	0.78484
w1	3.13666	3.38011E-5
p1	0.33135	0.0019
B	-0.74076	5.88177E-4
w2	2.82464	8.4184E-6
p2	-0.87829	5.92212E-4
C	-0.11176	3.09237E-4

From fitting

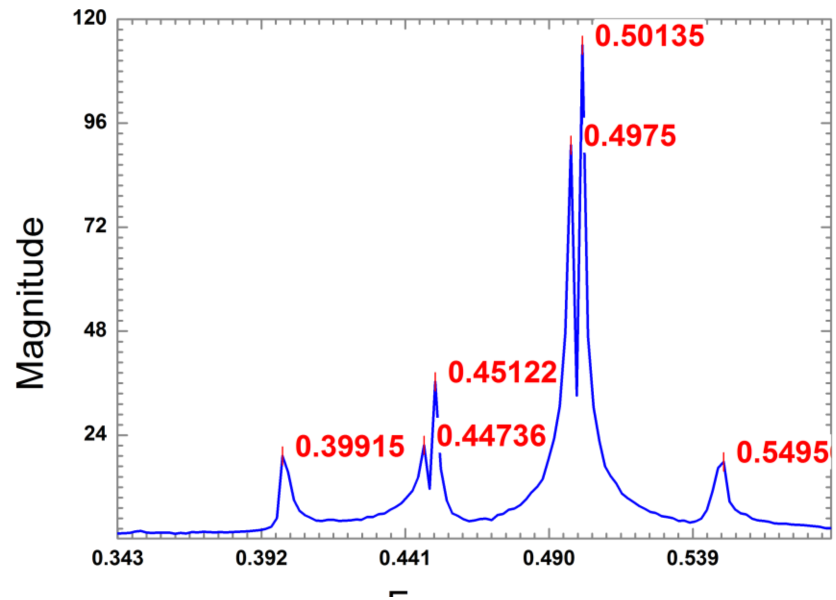
A	0.65012
t_0	199.64912
ω_1	3.13666
ϕ	0.33135
B	-0.74076
ω	2.82464
β	-0.87829
C	-0.11176

Result from FFT: ω_1=3.1402rad^{-1} and ω=2.8298 rad^{-1}

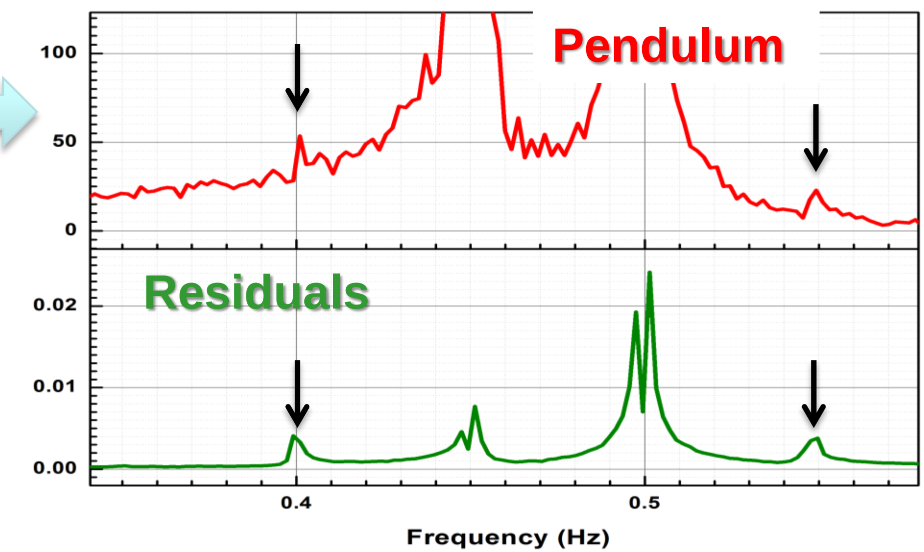
Beats. Experiment. Fitting. Residuals.



FFT



Compare with original pendulum spectrum



Possible origin of "extra" peaks:

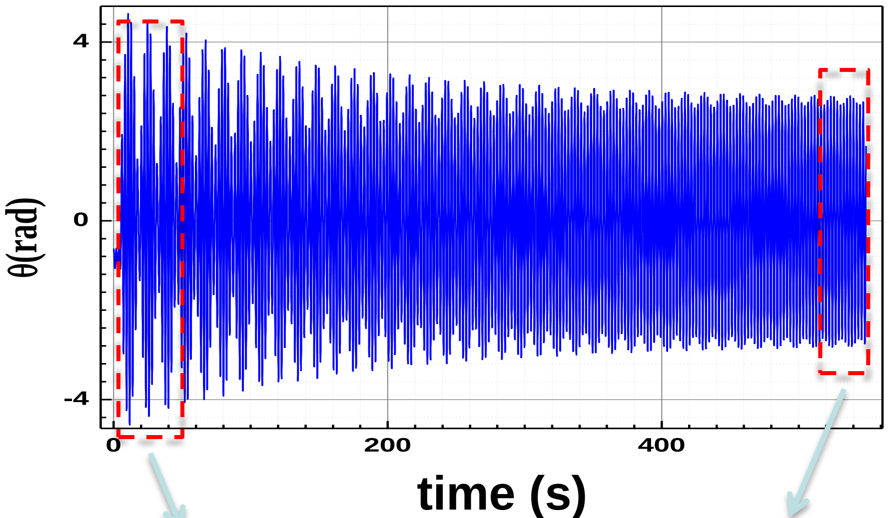
(i) Nonlinear behavior of pendulum

(ii) Not a single frequency driving force provided by motor

(iii) Not ideal fitting function

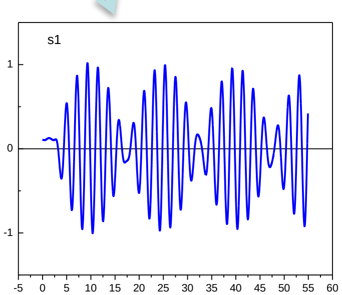
Beats. Experiment.

$$\theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at}\cos(\omega_1 t - \phi) + B\cos(\omega t - \beta(\omega))$$

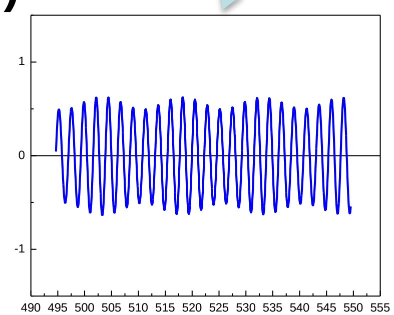


$\theta_t(t) \rightarrow 0$

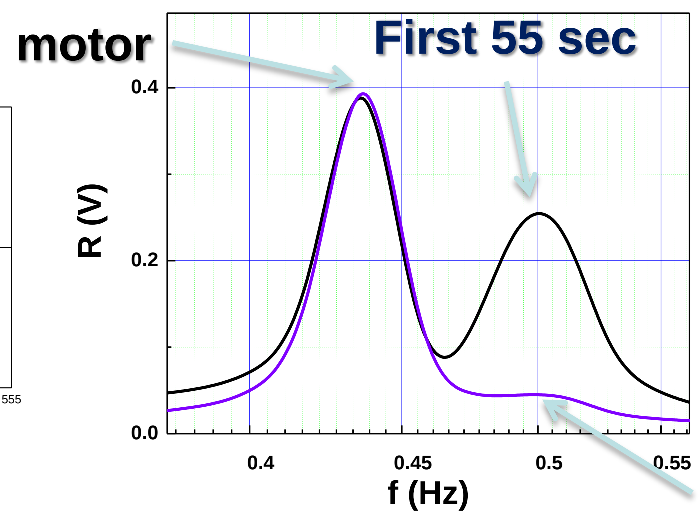
We also can analyze the decrease of the amplitude of the ω_1 component by analyzing the spectrum as a function of time



First 55 sec



Last 55 sec

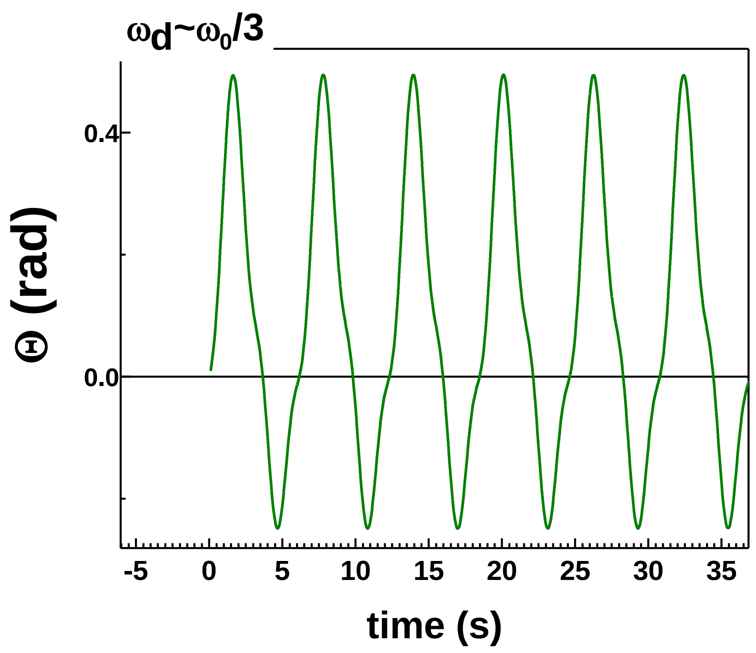


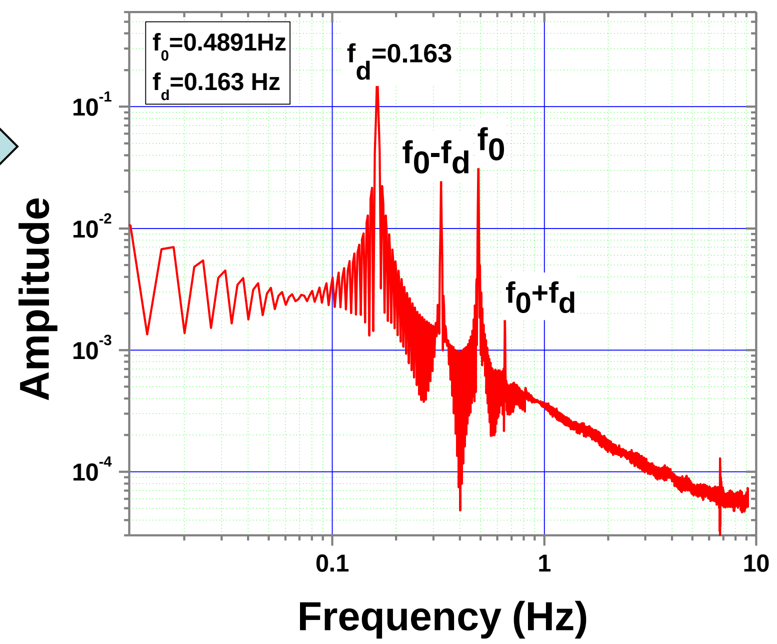
Last 55 sec

Origin 9.0: *Analysis → Signal Processing → FFT*

Beats. Experiment. More complicated case.

In the case of driving frequency f_d=f1/N where N is integer we can observe more complicated motion of the pendulum





Beats. Experiment. More complicated case.

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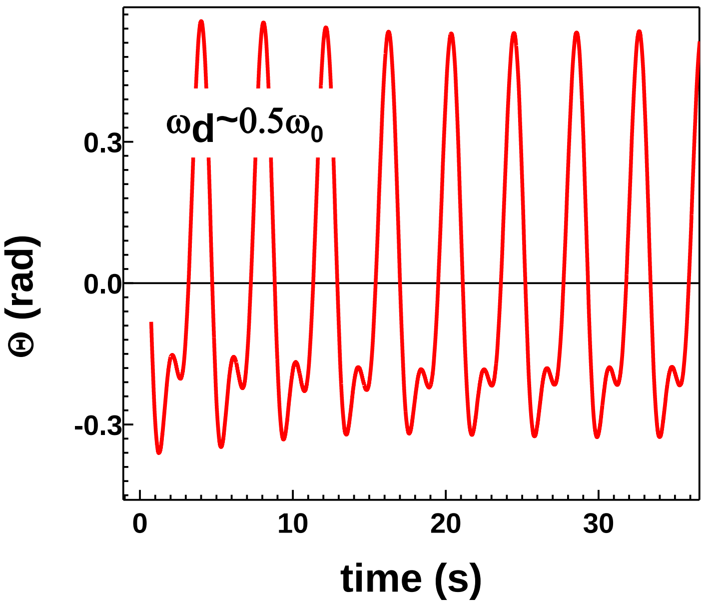
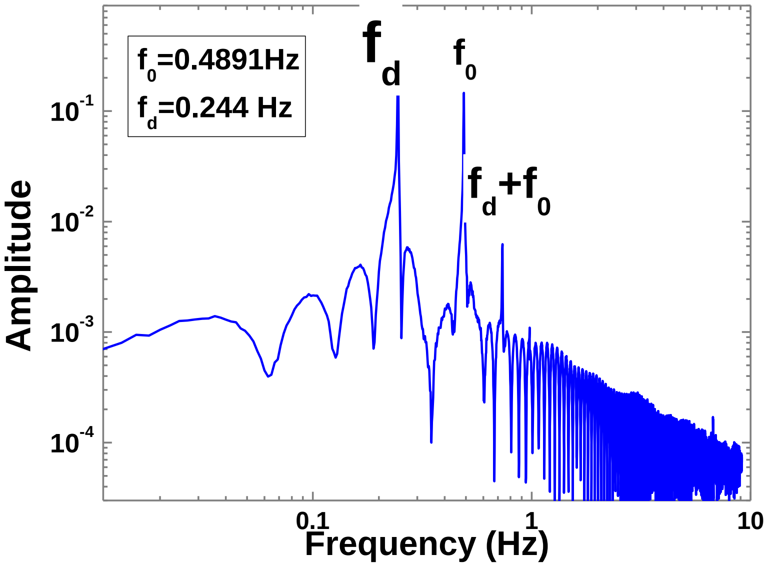
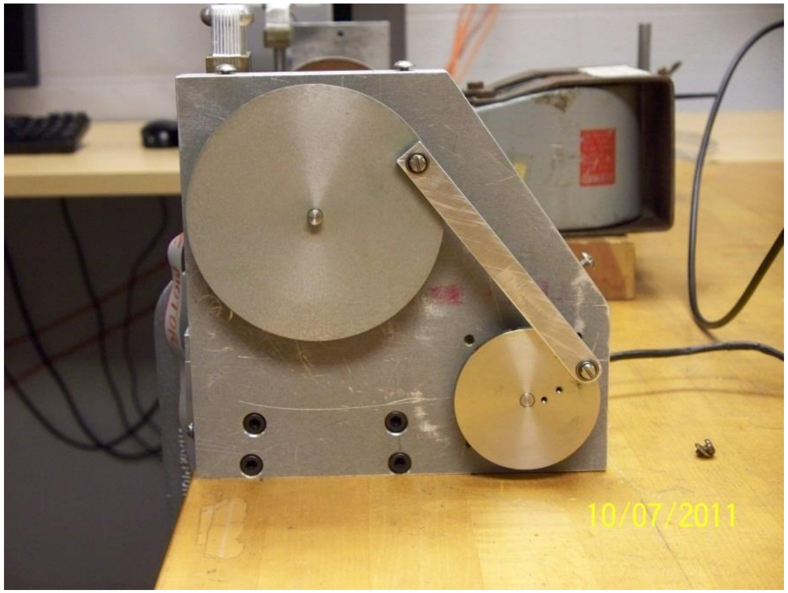
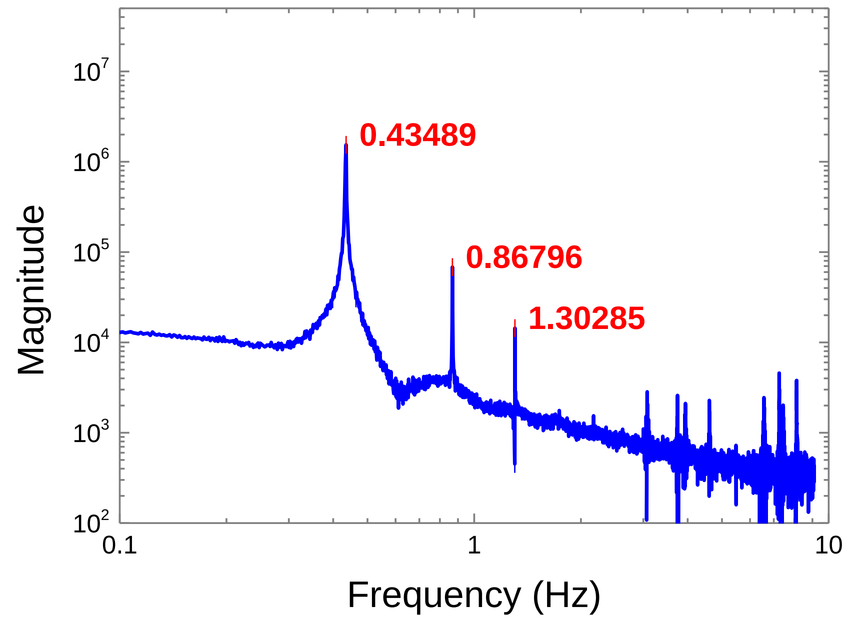
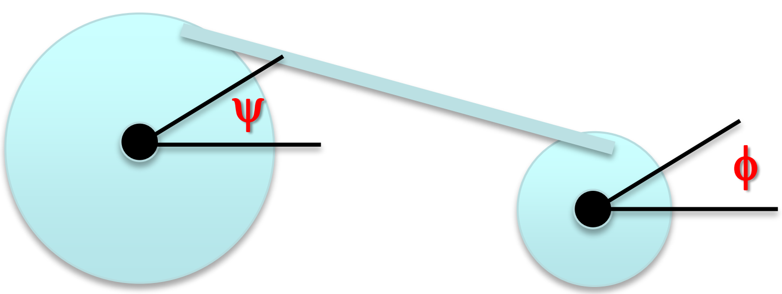




Beats. Experiment. Driving spectrum.



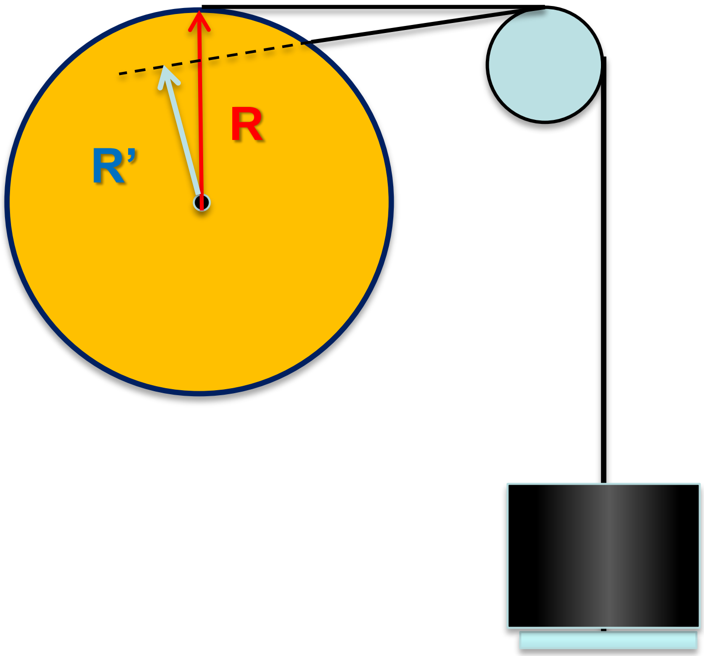


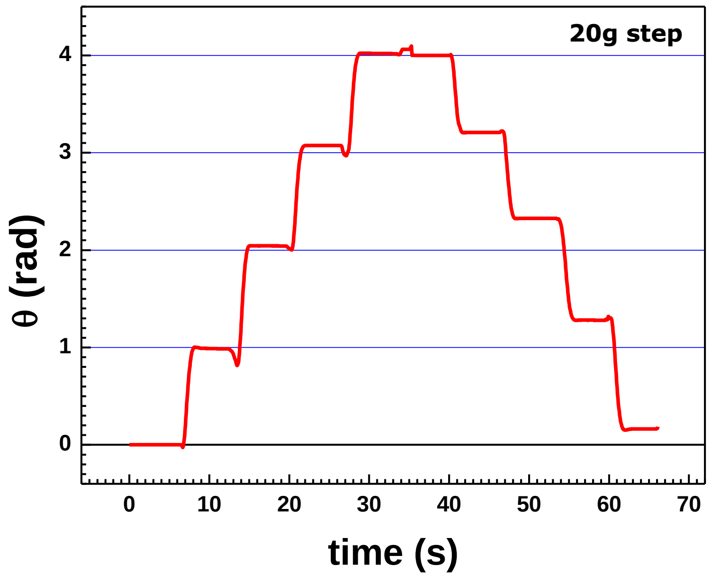


Detailed analyzes* shows that even if $\emptyset = \emptyset_0 \sin(\omega t)$ the driving torque contains several harmonics of ω

***P. Debevec (UIUC, Department of Physics)**

Comment on the previous Lab. Suggestions





$$\tau = R \times F$$

$$\theta = \frac{gR}{K} m$$